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Portfolio optimization with short-selling and spin-glass

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Abstract. In this paper, we solve a general problem of optimizing a portfolio in a futures markets framework, extending the previous work of Galluccio *et al.* [Physica A **259**, 449 (1998)]. We allow for long buying/short selling of a relatively large number of assets, assuming a fixed level of margin requirement. Because of non-linearity in the constraint, we derive a multiple equilibrium solution, in a size exponential respect to the number of assets. That means that we can not obtain the unique efficiency frontier, but many of them and each one is related to different levels of risk. Such a problem is analogous to that of finding the ground state in long-ranged Ising spin glass with external field. In order to get the best portfolio (*i.e.* that is along the best efficiency frontier), we have to implement a two-step procedure, performing the exhaustive enumeration of all local minima. We develop a concrete application, where the different part of the proposed solution are computed.

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1 Introduction

Portfolio theory is a basic pillar in economic analysis. It was originally proposed by Markovitz [1] during the 50's. The approach was that the return of any financial activity is described by a random variable, whose expected mean (measure for reward) and variance (interpreted as volatility) are assumed to be known from its historical past. The selection of a particular portfolio is based on the mean-variance principle: *i.e.*, if two portfolios are given, and the expected return of the first portfolio is higher than the second one, or the variance of the first portfolio is lower than the second one, we say that the first portfolio dominates the second; the latter being outside the decision field of a rational investor. Portfolio selection allows us to find the set of efficient portfolios, *i.e.* those portfolios not dominated by anything else. The rational investor eventually chooses among these efficient portfolios, in a subjective manner, according to his preferences towards risk. Much criticism, over the years, has been addressed to Markovitz's model. For instance, the choice of variance as a signal of risk has been criticized on logical grounds: the model requires a quadratic utility function and set of returns that must be normally distributed, but empirical analyses of financial price-data show that short time variations of the price of different assets deviate from the Gaussian distribution [2–4], which would be expected if agents were trading independently. Criticism has also been made of other aspects of the original model. In spite of this, the Markowitz approach has been very successful, because of its ability to grasp the hard core of the problem of how to allocate wealth among alternative assets. Extended versions of the model (for instance, in order to increase the number and types of assets considered; or to reduce computations for the estimated matrix of variance/covariances among returns), have almost continuously been introduced, and today various and updated versions of the original model are widely used by financial practitioners (for instance: [5,6]). Quoting from a successful fund management story[1]

> Even if point estimates of risk and return variables fail to represent reality fairly, insofar as inputs stem from well-grounded interrelationships, the mean-variance optimization process produces valuable insight into efficient portfolio alternatives.

A very interesting extension of the original model to the "short sales" case is due to the contribution of Lintner [7,8]. In fact, in this paper we move in the same direction extending and generalizing the traditional analysis (short-selling included), by considering the case of *futures markets*, where the short sale problem is regulated through the mechanism of margin accounts. Specifically – regarding the current state of the art in portfolio selection – the model for futures markets allows for:

– long-buying/short selling activities in equities,
– leveraging on margin accounts,
– a set of a (relatively) large number of assets to be prospected.

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[1] David Swensen, *Pioneering Portfolio Management: An Unconventional Approach to Institutional Investment* (Free Press, 1999).

The main result we aim to discuss here is that the portfolio optimization problem in futures markets naturally belongs to the realm of "complex" problems. In particular, a very large number of (quasi) equilibrium solutions coexists in the model and any procedure developed to reach a decision regarding the structure of the portfolio must face this problem. To be a little bit more precise, if we have N assets available to the investor, for a fixed expected return R, the number $n(N,R)$ of risk local minima grows exponentially in the number of assets, *i.e.*:

$$n(N,R) \sim \mathrm{e}^{\omega(R)N}$$

where $\omega(R)$ is a positive number depending on the portfolio return. Since we have this multiple equilibrium solution (however let us notice that the equilibria are not equivalent among themselves with respect to the level of risk), we have to implement a second step in the solution procedure in order to get the global equilibrium. This conclusion is a direct consequence of the application of Lagrange optimization and the *non-linear* constraint on the total wealth in futures markets. The issue of non-unique equilibrium is a well-known chapter of economic analysis, particularly in models with money, increasing returns and imperfect competition [9] and a new line of research on equilibrium beliefs is being developed. However it should be emphasized that, in this paper, the issue of multiple equilibrium solutions is discussed in a quite different setting. The underlying idea is that the selection of a *unique* optimum portfolio obtained from portfolio optimization in its current form [5,6], works well only if we introduce drastic and perhaps unrealistic simplification. Enlarging the picture to a certain extent, by relaxing the most restrictive assumptions in line with the practical experience of fund management, when calculating the solution for the optimum portfolio, we reach rapidly the threshold of a so-called NP problem, an area of research currently still largely ignored in economic computation [10,11]. We will deal in this paper with the complexity of the problem for a concrete case of a portfolio made up of 16 risky assets. We will explicitly show where complexity does arise and we will suggest the necessity of using algorithms typical of the realm of other optimization problems (like "simulated annealing" [12]).

Our analysis is built on a seminal idea by Galluccio *et al.* [13], who have re-stated the problem of a solution to the portfolio optimization problem in futures markets in terms of a spin glass problem (see also [14–16]). Motivated initially by the desire to understand the strange behaviour of certain magnetic alloys, the theory of spin glasses has provided a powerful paradigm for complex systems having competition and conflicting internal constraints. This is due to the fact that techniques developed for spin glasses have been successfully applied to other fields, like optimization problems, biology, social sciences [17,18]. In reference [13], Galluccio *et al.* showed that the problem of portfolio selection with short-selling, regulated through the mechanism of margin accounts, is basically equivalent to that of finding the ground state of a long-ranged Ising spin glass. The correlation matrix between assets return is related to the coupling matrix of the spin glass. Arguing that correlation matrix can be regarded as a generic realization of a matrix taken from an appropriate random ensemble, they derived the exponential increase of optimal solutions with the number of assets from the well-known increase of 1-flip-stable states with the number of spins [19,20]. Here we put forward their analysis, by explicitly constructing the *efficient frontier* of the portfolio. To achieve this aim, when performing the minimization of risk under the non-linear constraint, we need to fix the average return of the portfolio. In the spin glass language this is the analogous to imposing an external field [21].

The rest of the paper goes as follows. In the next Section we introduce the model of portfolio optimization in futures markets and we explain the non-linear constraint that is involved. A first analysis of the portfolio variance in the spirit of reference [13] is also presented, showing the complexity of the problem, compared to the classical Markowitz problem, where the wealth constraint is linear and one ends up always with a unique solutions. In Section 3 we generalize the optimization procedure by considering a fixed portfolio return. The general procedure and analytical calculation needed to construct the *efficient frontier* are obtained and similarities with the spin glass problem are enlightened. We deal with a concrete example in Section 4, considering a portfolio of 16 assets traded on the Nasdaq Market and solving the portfolio problem by means of computer calculations. We show the multiple equilibrium solutions and we discuss the distribution of local minimum risks. Then, we calculate the *efficient frontier* and discuss a related *averaged frontier*. This last curve, as discussed in the paper, represents the decision that an investor will *mostly likely* take by searching the optimal solution with standard methods of combinatorial optimization. We will also reconstruct and discuss the frontier corresponding to the "worst of the best decisions", namely the local minimum with higher risk (at fixed return). As we will see, the risk function has an exponential number of local minima and one needs to select by hand the lowest one. Moreover, there is the possibility that portfolios completely different among themselves correspond almost to the same value of the risk. Finally, in Section 5 some remarks on variances/covariances matrix are presented and summarized Conclusions will follow as usual.

2 The model

In this section, we present a model of portfolio optimization, where some traditional assumptions are relaxed. Specifically, in order to allow for the maximum flexibility in the model, a hedge-fund as rational investor agent is considered. Let us start from the standard definitions. We consider a portfolio $\mathcal{P}$ of N risky assets indexed by the subscript i which takes values $i = 1, \ldots, N$. The variance (*i.e.* risk) of the portfolio is

$$\sigma^2_{\mathcal{P}} = \sum_{i,j=1}^{N} C_{ij} p_i p_j = p^T C p \qquad (1)$$

and the mean (*i.e.* expected average return)

$$R_{\mathcal{P}} = \sum_{i=1}^{N} p_i r_i = p^T r \tag{2}$$

where $p = (p_1, p_2, \ldots, p_N)$ are the shares of a given total wealth to be invested, and the external parameters are

r_i – the expected return on asset i,

C_{ij} – the matrix of variances ($i = j$) and covariances ($i \neq j$).

and we have introduced the usual vectorial notation with ^{T} to denote the transposed.

The aim of the optimization is to find the most efficient investment strategy, *i.e.* to evaluate proportions p of the total wealth W that minimize the risk $\sigma^2_{\mathcal{P}}$, for a given return $R_{\mathcal{P}}$ (or *vice versa* shares p that maximize the return $R_{\mathcal{P}}$ for a given level of risk $\sigma^2_{\mathcal{P}}$)). In other words, we would like to calculate the *efficient frontier*, that represents the relationship between the risk of the portfolio and the expected return of the portfolio itself having the best utility for the investor. Knowing the efficient frontier, once a particular expected rate of return has been identified, we can determine the correspondent efficient portfolio; it is such that its variance (or riskiness) is a minimum. Therefore, once one of the two (either the risk or the return) has been chosen by the investor, the other variable is derived as a consequence.

Since long-buying and short-selling are allowed, and leveraging on margin accounts works, then the budget constraint is

$$\sum_{i=1}^{N} \Gamma \mid p_i \mid = W \tag{3}$$

where Γ is the (fixed) margin constraint and $p_i > 0$ or $p_i < 0$, depending on the sign of the contract (buy or sell respectively). The margin is assumed fixed for all operations, and it does not change over time according to price variations of the underlying assets. Moreover, the problem of issuing futures on behalf of the financial institution is not considered. Without loss of generality we can set $W/\Gamma = 1$, so that, introducing the vector s whose components are $s_i = \mathrm{sign}(p_i)$, the budget constrain becomes:

$$\sum_{i=1}^{N} \mid p_i \mid = p^T s = 1 \tag{4}$$

where sign denotes the sign function, $\mathrm{sign}(x) = 1$ if $x > 0$ and $\mathrm{sign}(x) = -1$ if $x < 0$.

2.1 "Complexity" of the model

Let us consider the problem of finding the minimum of the variance subjected to the only budget constraint (no fixed average portfolio return). In other words we want to minimize the portfolio variance equation (1) with the non-linear constrain given by equation (4). We introduce a Lagrangian function with one Lagrange multiplier μ:

$$L(p, \mu) = p^T C p - \mu(p^T s - 1). \tag{5}$$

Differentiating with respect to the $N + 1$ unknowns p and μ we obtain the following equations for the extreme points

$$p = \frac{1}{2}\mu C^{-1} s \tag{6}$$

$$p^T s = 1 \tag{7}$$

where C^{-1} is the inverse of the correlations matrix. Inserting equation (6) in equation (7) we can solve for μ and then for p

$$\mu = \frac{2}{s^T C^{-1} s} \tag{8}$$

$$p = \frac{1}{s^T C^{-1} s} C^{-1} s. \tag{9}$$

Applying the sign function to both sides of the last equation, we obtain

$$s = \mathrm{sign}(C^{-1} s) \tag{10}$$

where we have used the fact that, since C is a positive definite matrix, the same is true for C^{-1}, so that $s^T C^{-1} s > 0$ for every value of s.

The original problem has thus been mapped in finding the solution of equation (10): once the s_i that solve this equation are known, the shares p_i can be calculated using equation (9), while the portfolio variance is given by

$$\sigma^2_{\mathcal{P}} = \frac{1}{s^T C^{-1} s} \cdot \tag{11}$$

But solving equation (10) is a very tough task. It is exactly the same equation that appears in spin glass theory when one looks for 1-flip stable configurations at zero temperature. It is well-known [19,20] that equation (10) admits for a generic random matrix C^{-1} an exponential number of solution. Moreover these solutions are "chaotic", *i.e.* they are completely different one from another and they completely change varying the number of degree of freedom. In our case the matrix C^{-1} is not *a priori* random but it is constructed from the historical dates. Nevertheless, since historical prices/returns movements are generated by market fluctuations, the correlation matrix C (and so its inverse C^{-1}) can be seen as a generic realization of some specific random matrix ensemble (see Sect. 5). In this way we can borrow the results from physics and directly draw some first conclusions [13]:

- At variance with the classical Markowitz portfolio problem, where we always find a minimization equation that admits a unique solution, in the present case of futures markets we have an exponential number of portfolios for which the risk function has a (local) minimum. So we face the embarrassment of which solution to choose and we need to calculate by hand the portfolio variance on each solution to find the true minimum.
- We can have very different portfolios corresponding to (local) risk minima having almost the same risk value.
- Adding one asset to the portfolio radically changes the shape of efficient investments.

3 Constructing the efficient frontier

In the previous section we have shown how the complexity of the problem naturally arises in the minimization procedure for the case of the global minimum of the portfolio risk. This says nothing about the efficient frontier (even if it shows the instability of rational investment decisions). To completely solve the problem, we have to repeat the minimization of the variance fixing the average return to the value R with an extra Lagrange multiplier ν.

Thus the problem is now to minimize equation (1) subject to equation (4) and to the additional constraint

$$R_{\mathcal{P}} = p^T r = R. \tag{12}$$

We introduce the Lagrangian function

$$L(p,\mu) = p^T C p - \mu(p^T s - 1) - \nu(p^T r - R). \tag{13}$$

Differentiating with respect to the $N+2$ unknowns p, μ and ν we obtain

$$\begin{aligned} p &= \frac{1}{2}\mu C^{-1} s + \frac{1}{2}\nu C^{-1} r \\ p^T s &= 1 \\ p^T r &= R. \end{aligned} \tag{14}$$

Inserting the first equation in the second and the third, we can solve for μ and ν, and then for p. Defining

$$\begin{aligned} \alpha &= s^T C^{-1} s \\ \beta &= r^T C^{-1} s \\ \gamma &= r^T C^{-1} r \end{aligned} \tag{15}$$

we obtain the following expressions:

$$\begin{aligned} \mu &= 2\frac{\gamma - R\beta}{\alpha\gamma - \beta^2} \\ \nu &= 2\frac{R\alpha - \beta}{\alpha\gamma - \beta^2} \\ p &= \frac{\gamma - R\beta}{\alpha\gamma - \beta^2} C^{-1} s + \frac{R\alpha - \beta}{\alpha\gamma - \beta^2} C^{-1} r. \end{aligned} \tag{16}$$

Applying the sign function to both sides of the last equation and remembering that $s = \text{sign}(p)$ by definition, we finally obtain

$$s = \text{sign}\left(\frac{\gamma - R\beta}{\alpha\gamma - \beta^2} C^{-1} s + \frac{R\alpha - \beta}{\alpha\gamma - \beta^2} C^{-1} r\right) \cdot \tag{17}$$

This is the basic equation that substitute equation (10) in the case of a fixed average return R. If we now identify the

$$\frac{\gamma - R\beta}{\alpha\gamma - \beta^2} C^{-1} \Longleftrightarrow J$$

$$\frac{R\alpha - \beta}{\alpha\gamma - \beta^2} C^{-1} r \Longleftrightarrow h$$

we establish a perfect analogy between the equation to be solved for the portfolio optimization problem in futures markets and the one for the spin glass problem of finding the local energy minima at zero temperature in presence of an external field h. Note that in equation (17) the coupling matrix depends on the spin configuration itself.

The general procedure for tracing the N-stocks efficient frontier in the case of *futures markets* thus can be summarized as follows:

1. Fix a certain value of the average expected portfolio return R.
2. For this return R solve the system of N equations (17) for the vector $s = (s_1, s_2, \ldots, s_N)$. In general the number of solutions n will be exponential in number of assets N: $n \sim \mathrm{e}^{\omega N}$, where the exponential rate $\omega = \omega(R)$ depends on the fixed return R.
3. Calculate the value of the proportions investment $p = (p_1, p_2, \ldots, p_N)$ corresponding to each solution of step 2 through formula (16) and then the associated risk. Select the lowest value of the risk and the corresponding optimum portfolio investment.
4. Increase the return R by a certain (constant) amount and repeat the entire procedure from step 2 through 4.

4 A worked example

In this section we explicitly treat an example with real data: we demonstrate the "complexity" of our problem and we calculate the *efficient frontier* by means of computer calculations.

4.1 Data

We considered the case of a portfolio consisting of $N = 16$ risky assets. These risky assets are some common stocks traded on the Nasdaq, in the period October 1, 1998 – November 13, 2000. A historical record of daily prices of these stocks for the $T = 553$ trading days of the period was used to estimate the relevant parameters – the mean return r_i and the variance/covariance matrix C_{ij}. The data source is *DataStream*. Calling $x(i,k)$ the price of the ith asset (where $i = 1, \ldots, N$) at the kth day (where $k = 1, \ldots, T$), the daily rates of return are:

$$r(i,k) = \frac{x(i,k+1) - x(i,k)}{x(i,k)} \tag{18}$$

while the formula used to estimate average returns and covariances are:

$$r_i = \frac{1}{T-1}\sum_{k=1}^{T-1} r(i,k) \tag{19}$$

$$C_{ij} = \frac{1}{T-1}\sum_{k=1}^{T-1} \left[r(i,k) - r_i\right]\left[r(j,k) - r_j\right]. \tag{20}$$

The estimated mean returns are given in Table 1, which also lists the stocks by name, while the variance/covariance matrix is split in Tables 2 and 3.

Table 2. Variance/covariance matrix.

	ADO	AMA	AME	AOL	APP	BRO	CIS	CMG
ADO	0.0017655	0.0006738	0.0005875	0.0004569	0.0010777	0.0007906	0.0006333	0.0009492
AMA	0.0006738	0.0039831	0.0015211	0.0014186	0.0009082	0.0013937	0.0008632	0.0025806
AME	0.0005875	0.0015211	0.0059604	0.0014733	0.0009082	0.0013879	0.0007159	0.0021351
AOL	0.0004569	0.0014186	0.0014733	0.0018082	0.0006482	0.0010780	0.0006245	0.0017036
APP	0.0010777	0.0009082	0.0009082	0.0006482	0.0038179	0.0016104	0.0011169	0.0017814
BRO	0.0007906	0.0013937	0.0013879	0.0010780	0.0016104	0.0052910	0.0008336	0.0020747
CIS	0.0006333	0.0008632	0.0007159	0.0006245	0.0011169	0.0008336	0.0011323	0.0011854
CMG	0.0009492	0.0025806	0.0021351	0.0017036	0.0017814	0.0020747	0.0011854	0.0051837
DEL	0.0004815	0.0006627	0.0006693	0.0006179	0.0006952	0.0006867	0.0007166	0.0011147
DOU	0.0006174	0.0019525	0.0022296	0.0014429	0.0013858	0.0020133	0.0007361	0.0029053
EBA	0.0007485	0.0022906	0.0016129	0.0013104	0.0012851	0.0017050	0.0009150	0.0025681
INK	0.0006859	0.0019766	0.0015506	0.0012455	0.0013270	0.0020950	0.0009573	0.0026140
INT	0.0005465	0.0006466	0.0005004	0.0005156	0.0009470	0.0006798	0.0006638	0.0009792
JDS	0.0007854	0.0008836	0.0009315	0.0007894	0.0016879	0.0013783	0.0010121	0.0015445
MIC	0.0003891	0.0005764	0.0004947	0.0004240	0.0005583	0.0006681	0.0004818	0.0007499
ORA	0.0007064	0.0008636	0.0007249	0.0005809	0.0011963	0.0009879	0.0007990	0.0012780

Table 1. Rate of return.

	Name	*Symbol*	*Return*
1	Adobe	ADO	0.004818
2	Amazon	AMA	0.002838
3	Ameritrade	AME	0.005522
4	Aol	AOL	0.003374
5	Appmc	APP	0.008313
6	Broadvis	BRO	0.008397
7	Cisco	CIS	0.002844
8	Cmgi	CMG	0.004129
9	Dell	DEL	0.000263
10	Doubleclick	DOU	0.004788
11	Ebay	EBA	0.005765
12	Inktomy	INK	0.004144
13	Intel	INT	0.001683
14	Jdsuni	JDS	0.005999
15	Microsoft	MIC	0.000843
16	Oracle	ORA	0.004045
**	Nasdaq Index	NAS	0.001362

4.2 Multiple solutions

First of all we analyzed the whole set of possible choices that one obtains for a fixed value of the return on the portfolio composed by $N = 16$ assets. We chose to fix the portfolio return to the value of the average daily Nasdaq index return in the period we considered: $R = R_{\text{NAS}} = 0.0014$ (*i.e.* 0.14%). Applying the solving technique described in the previous Section, we found the solutions $\hat{s} = (\hat{s}_1, \hat{s}_2, \dots, \hat{s}_N)$ of the equation (17) doing an exhaustive enumeration of all the 2^N possible values of the N-dimensional vector s. We found 6675 $\hat{s}^{(j)}$ vectors that satisfy equation (17). From them we calculate the corresponding proportions $\hat{p}^{(j)}$ using equation (16) and the risk values $\hat{\sigma}^{(j)}$ using equation (1). Here the index j labels all the local risk minima, $j = 1, 2, \dots, 6675$. The risk ranges between $\hat{\sigma}_{\text{MIN}} \approx 0.0090$ and $\hat{\sigma}_{\text{MAX}} \approx 0.0162$ and has an *averaged risk* $\hat{\sigma}_{\text{AVE}} \approx 0.0100$, where the averaged value is obviously defined as

$$\hat{\sigma}_{\text{AVE}} = \frac{1}{6675} \sum_{j=1}^{6675} \hat{\sigma}^{(j)}.$$

We show how the risk values are distributed in Figure 1, where we plot their probability density function, *i.e.* the histogram normalized to have area 1. We see that most of the risk local minima have a risk value higher than the global risk minimum $\hat{\sigma}_{\text{MIN}}$.

It is important to remark that the local minima which correspond to the maximum of the distribution shown in Figure 1 are in fact the solutions that one would obtain with very "high probability". To be more precise, if one considers portfolios of some bigger dimension, just say 100 assets, it would be impossible to compute exactly all the local minimum and instead one should rely on other methods. For example (but not only) one could implement an algorithm developed in the realm of spin glass theory, aimed to freeze spin glass systems down to zero temperature in order to reach the real minimum (ground state). This rich class of algorithms can be roughly speaking addressed as *simulated annealing* techniques [12,17]. If one tries to implement these algorithms then it would be almost impossible to end up on the efficient frontier, instead one would almost surely converge to a portfolio solution corresponding to the average risk (see also Sect. 4.4).

Moreover we observe that the proportions $\hat{p}$ can be very different among themselves and from $\hat{p}_{\text{MIN}}$. This is shown in Table 4 where are reported the portfolio p_{MIN} (first column) corresponding to $\hat{\sigma}_{\text{MIN}}$, the portfolio $\hat{p}_{\text{MAX}}$ (last column) corresponding to $\hat{\sigma}_{\text{MAX}}$ and some portfolios $\hat{p}$ (middle columns) having a risk value around $\hat{\sigma}_{\text{AVE}}$. From

Table 3. Variance/covariance matrix (Continued).

	DEL	DOU	EBA	INK	INT	JDS	MIC	ORA
ADO	0.0004815	0.0006174	0.0007485	0.0006859	0.0005465	0.0007854	0.0003891	0.0007064
AMA	0.0006627	0.0019525	0.0022906	0.0019766	0.0006466	0.0008836	0.0005764	0.0008636
AME	0.0006693	0.0022296	0.0016129	0.0015506	0.0005004	0.0009315	0.0004947	0.0007249
AOL	0.0006179	0.0014429	0.0013104	0.0012455	0.0005156	0.0007894	0.0004240	0.0005809
APP	0.0006952	0.0013858	0.0012851	0.0013270	0.0009470	0.0016879	0.0005583	0.0011963
BRO	0.0006867	0.0020133	0.0017050	0.0020950	0.0006798	0.0013783	0.0006681	0.0009879
CIS	0.0007166	0.0007361	0.0009150	0.0009573	0.0006638	0.0010121	0.0004818	0.0007990
CMG	0.0011147	0.0029053	0.0025681	0.0026140	0.0009792	0.0015445	0.0007499	0.0012780
DEL	0.0014036	0.0007637	0.0008658	0.0008183	0.0007496	0.0006910	0.0005324	0.0005532
DOU	0.0007637	0.0057720	0.0018463	0.0020844	0.0006473	0.0009491	0.0006221	0.0006542
EBA	0.0008658	0.0018463	0.0050324	0.0023282	0.0006089	0.0010614	0.0005742	0.0008820
INK	0.0008183	0.0020844	0.0023282	0.0046071	0.0007030	0.0012190	0.0005463	0.0009930
INT	0.0007496	0.0006473	0.0006089	0.0007030	0.0011612	0.0008158	0.0004691	0.0006545
JDS	0.0006910	0.0009491	0.0010614	0.0012190	0.0008158	0.0025830	0.0005088	0.0010459
MIC	0.0005324	0.0006221	0.0005742	0.0005463	0.0004691	0.0005088	0.0008008	0.0004617
ORA	0.0005532	0.0006542	0.0008820	0.0009930	0.0006545	0.0010459	0.0004617	0.0019352

Table 4. Some portfolios corresponding to fixed return $R = R_{\mathrm{NAS}} = 0.0014$. The first column is the "best" portfolio corresponding to the global risk minimum $\hat{\sigma}_{\mathrm{MIN}} \approx 0.0090$. The last column is the "worst" portfolio corresponding to the highest of the local risk minima $\hat{\sigma}_{\mathrm{MAX}} \approx 0.0162$. In the middle columns there are some portfolios having risk around the average risk value $\hat{\sigma}_{\mathrm{AVE}} \approx 0.0100$.

Risk	0.0090	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100	0.0162
ADO	0.0630	−0.0070	0.0780	0.0890	0.1070	0.0990	0.1100	0.0720
AMA	−0.0460	−0.0520	−0.0390	−0.0260	−0.0320	−0.0380	−0.0080	0.0250
AME	0.0150	0.0210	0.0330	−0.0020	−0.0160	0.0170	0.0330	0.0310
AOL	0.0820	0.0950	−0.0280	−0.0110	0.1480	0.0900	−0.0180	0.1250
APP	0.0320	0.0600	0.0380	0.0480	0.0660	0.0800	0.0720	0.0280
BRO	0.0390	0.0340	0.0340	0.0260	0.0360	0.0350	0.0410	0.0550
CIS	0.1230	−0.1610	−0.1770	−0.1570	−0.1280	−0.1720	−0.1460	0.0670
CMG	−0.0450	0.0030	0.0170	−0.0390	−0.0330	0.0020	−0.0440	0.0120
DEL	−0.1380	−0.1600	−0.1520	−0.0800	−0.0390	0.0410	0.1070	0.0000
DOU	0.0220	−0.0230	−0.0170	0.0320	−0.0150	−0.0280	−0.0100	−0.1100
EBA	0.0410	0.0490	0.0520	0.0510	0.0460	0.0320	0.0340	0.0420
INK	−0.0300	−0.0320	−0.0290	−0.0250	−0.0280	−0.0300	0.0180	−0.1640
INT	0.0890	0.0970	0.1010	−0.1090	−0.0760	−0.1470	−0.1140	0.0560
JDS	−0.0440	0.0580	0.0630	0.0870	0.0860	0.0820	0.0930	0.0100
MIC	−0.1490	0.0900	0.0950	0.1400	−0.1080	0.0800	−0.1360	0.1720
ORA	0.0420	0.0570	0.0470	0.0760	−0.0370	−0.0260	−0.0170	0.0300

a practical point of view, this *quasi-degeneracy of the risk value with corresponding proportions $\hat{p}$ very far from each other* is the most interesting finding of this paper. In order to illustrate better this point, we use a very important and not trivial object in spin glass theory, the *overlap distribution.* Let us denote the vector sign of the global minimum solution $\bar{s} = \{\bar{s}_1, \ldots, \bar{s}_N\}$ and by $\hat{s} = \{\hat{s}_1, \ldots, \hat{s}_N\}$ another generic local minimum solution (for simplicity, we consider only the sign of the proportions $s_i = \mathrm{sign}(p_i)$, but the same reasoning can be extended to the proportions themselves). A simple number, which describe how different these two solutions are, can be defined basically by counting how many s_j's one must flip in order to go from one configuration to the other. This information is contained in the integer number $m(\bar{s}, \hat{s})$ defined as:

$$m(\bar{s}, \hat{s}) = \frac{1}{2}\left(N - \sum_{k=1}^{N} \bar{s}_k \hat{s}_k\right)$$

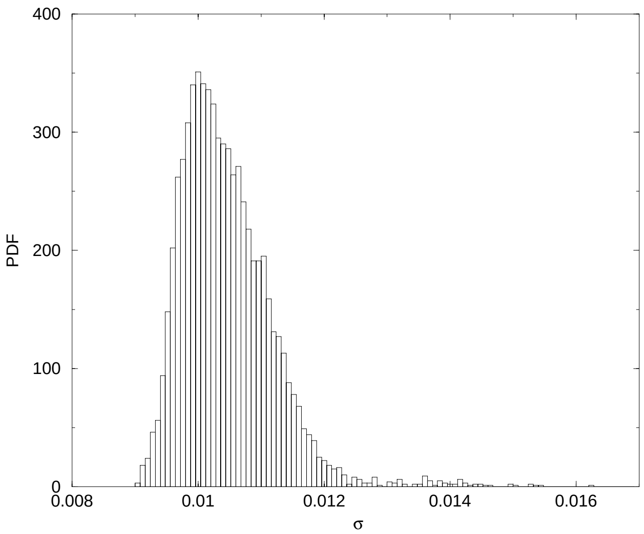

Fig. 1. Histogram of the local risk minima. The number of assets is $N = 16$ and the fixed portfolio return is the Nasdaq return $R = R_{\rm NAS} = 0.0014$. *PDF* stands for Probability Density Function.

and it ranges from 0 (identical sign vectors) to N (opposite sign vectors).

In the top side of Figure 2 we plot the probability distribution function of the numbers m_j's, obtained by measuring the number of different signs between any single local minimum and the global solution: $m_j = m(\bar{s}, \hat{s}^{(j)})$, where $\hat{s}^{(j)}$ runs over all possible 6675 solutions. In particular, this histogram show clearly that *most of* the solutions are in turn very different from the one we would like to calculate *a priori*, *i.e.* $\bar{s}$.

Actually, a little bit more than this can be said. In general two different local solutions might have almost the same risk level but in a strategic-economic context they can be totally different. This is shown in the bottom side of Figure 2, where the histogram of the overlaps between all possible solutions is shown. Summarizing, a multiple choice is available to the investor and an irreducible component of arbitrariness is present in the final decision. Moreover, it is likely that traditional method for reconstructing the minimizing solution will lead the investor to be "trapped" into a quite different local minimum.

4.3 Exponential growth of solutions in the number of assets

The impossibility of taking a rational decision described in the previous paragraph gets even worse if we consider the dependence of the number of solutions on the number of assets. We performed numerical experiments varying N from 5 to 16 keeping the average return R fixed to $R_{\rm NAS}$. For each value we calculate the number of local risk minima $n(N)$, *i.e.* the number of solutions of equation (17). As it is clear from Figure 3 this number grows exponentially with the number of assets (note the

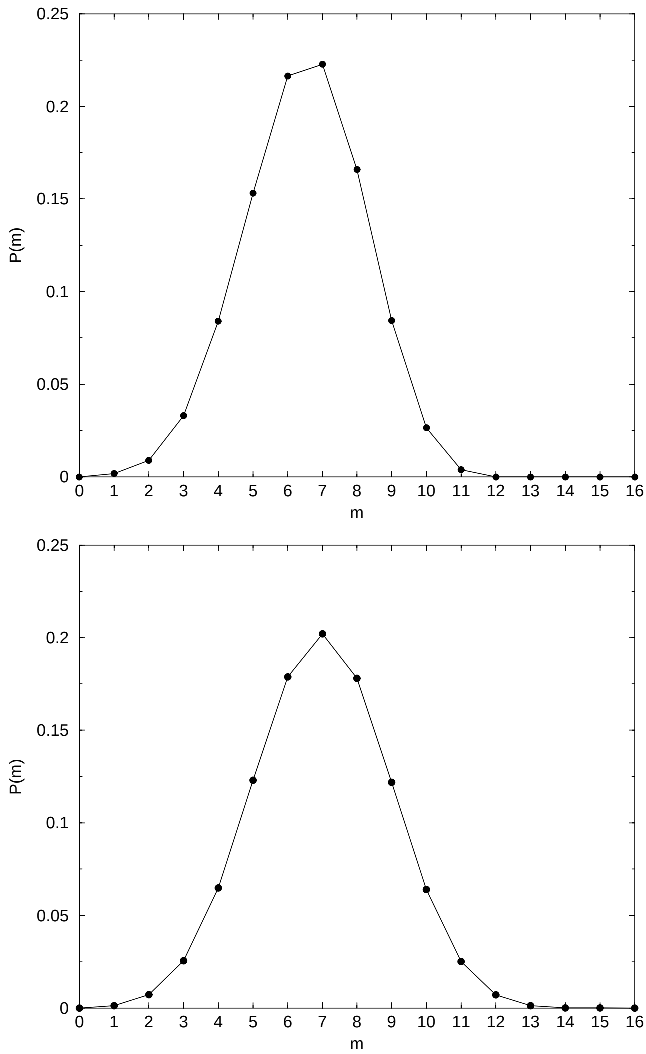

Fig. 2. Top: the distribution of the overlap between the configuration $\bar{s}$ corresponding to the risk global minimum and the other configurations $\hat{s}^{(j)}$ corresponding to the risk local minima. Bottom: the distribution of the overlap between all the configuration $\hat{s}^{(j)}$ (including $\bar{s}$) corresponding to the risk local minima.

lin-log scale in the graph). The best numerical fit yields $n(N) \sim \exp(0.69N)$. We note that essentially the same value $n_{NY}(N) \sim \exp(0.68N)$ has been found in reference [13] for the case of 20 assets of the New York Stock Exchange with no fixed portfolio return.

This exponential growth of the number of risk local minima with the number of assets is the exact analog of the exponential growth of the number of local energy minima typical of spin glasses [19,20] From the economic point of view, the consequence of this growth is that the arbitrariness degree enlightened above gets even bigger by increasing the portfolio dimension. Moreover we have verified (we do not report data for the sake of space) that the multiple solutions have a "chaoticity" property, in the sense that a small change of the correlation matrix C, or the addition of an extra asset, completely changes the

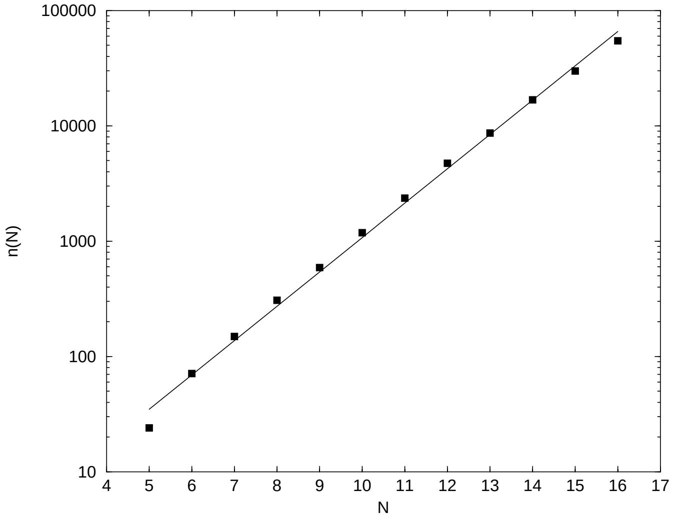

Fig. 3. The number of local risk minima *versus* the number of assets N. The fixed average return is the Nasdaq return $R_{\rm NAS} = 0.0014$. The solid line is the best numerical fit $n(N) \sim \exp(0.69N)$.

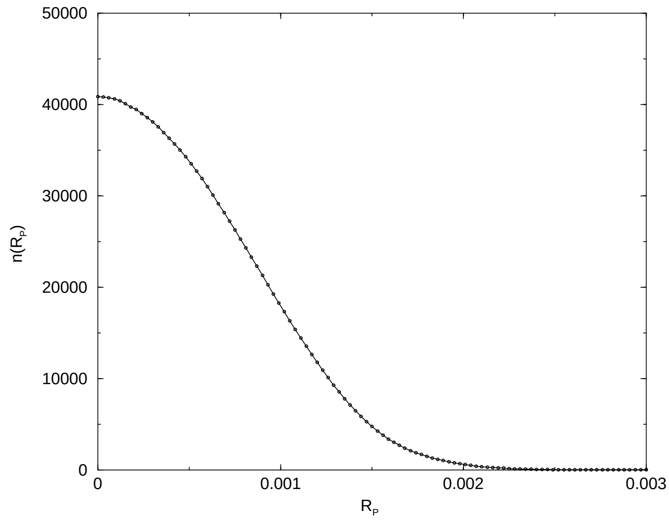

Fig. 4. The number of local risk minima *versus* the average portfolio return. The number of assets is fixed to $N = 16$.

values of optimal proportions. On the other hand the number of possible decisions decreases for increasing value of the fixed return R under which minimization is performed. We argue this fixing $N = 16$ and calculating the number of risk local minima varying the return R in the range $[0, 0.003]$. In Figure 4 we plot the number of solutions of equation (17) corresponding to each R value. We see that the local risk minimums decrease for increasing value of the return, becoming zero at $R \sim 0.0026$. Above this threshold we do not find from equation (17) any portfolios satisfying both the budget and the return constraints. One should look for risk minima on the border of the manifold where Lagrange optimization is performed. We did not investigate this point further.

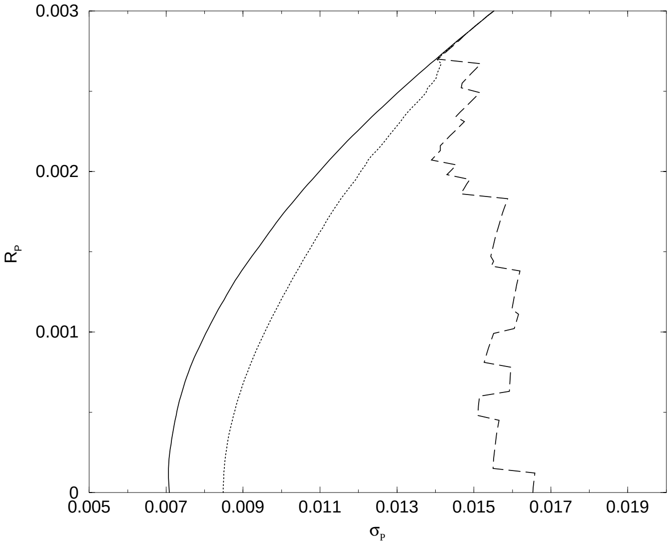

Fig. 5. Efficient frontiers for the portfolio consisting of 16 assets. The continuous line is the *"best-efficient"* frontier corresponding to the lowest value between all the risk local minima. The dotted line shows the *"average-efficient"* frontier, *i.e.* it corresponds to the average value of the risk between all the risk local minima. The dashed line is the *"worst-efficient"* frontier, constructed by using the higher values of the risk local minima for each fixed return.

4.4 The efficient frontier

Here we use the data concerning the 16-stocks in order to compute the efficient frontier and two more related curves. More precisely, we first allow the averaged return to range from 0 to 0.003 in 100 constant steps and for each value of the return we calculated the proportions p corresponding to the risk local minima. Then, among them we selected the one associated to the risk global minimum and the one associated to the worst choice, namely the local minimum corresponding to the portfolio at the very right tail of Figure 1. Moreover, for a given fixed return, we also calculate the *averaged risk* $\sigma_{\rm AVE}$. Namely, if n is the total number of local minima for a fixed return R and $\sigma^{(j)}$ are the associated risks $(j = 1, \ldots, n)$, we define

$$\sigma_{\rm AVE} = \frac{1}{n}\sum_{j=1}^{n}\sigma^{(j)}.$$

By varying the return R, we use this data to reproduce a kind of *averaged* efficient frontier, which is shown in Figure 5, together with the other two. We checked that, as one could expect, the proportion of the investment associated to the smallest and greatest local minima are completely different. Furthermore also the portfolios corresponding to risk value around the average risk $\sigma_{\rm AVE}$ are very different, yielding many different equivalent investment strategies.

5 A few remarks on the variances/covariances matrix

A few words of comment about the correlation matrix are in place. The computation of matrix C of variances/covariances between return variations of different assets is a key issue in the theoretical solution of the portfolio optimization problem. The study of these correlation matrices is a crucial problem in financial theory, both for developments of Markowitz's theory of optimal portfolios, and for problems (more recently introduced) of risk management, related to the so-called value at risk models. For covariance matrices used in value at risk models, the discussion is open to an almost continuous flow of new experiments, with an empirical evaluation of the obtained results, based on a single case (for instance see [22]).

In reference [13] a random matrix approach is proposed by Gallucio *et al.* as an alternative to well-established index models, originally presented by Markowitz himself and developed by Sharpe [5]. These authors assume the matrix C of variance/covariance as random; in particular, from an historical analysis on asset prizes variations in various Stock Markets, they argue that the correlation matrix can be well approximated by a matrix which is a generic realization of the so-called Exponential Orthogonal Ensemble. In this way they can exploit a consolidated approach based on random matrix theory [23] and use self-averaging property to prove analytically the exponential increase of risk local minima. Assuming the matrix C as random is quite reasonable in terms of financial theory: prices/returns movements are in any case random, since they can be read as realization of a stochastic process, generated by market fluctuations. The analysis has successively been enforced in a series of papers [24–27] by examining the eigenvalues and eigenvectors distribution of the matrix itself. The main aim here is, to separate the randomness contained in the data from the *real market* information. Let us consider a portfolio of N assets: the correlation matrix contains $N(N-1)/2$ entries, which must be computed from N time-series of length T. If T is small compared to N, one would expect that the determination of the covariance is most likely to be noisy, and therefore the empirical correlation matrix is to a large extent random; this implies that the structure of the matrix is dominated by measurement noise and real information are somewhat hidden in the data. A deep analysis of eigenvalues (and corresponding eigenvectors) performed in references [24,25] sheds relevant light on the statistical properties of empirical correlation matrices. In the case of the S & P500, less than 6% of the eigenvectors, which are responsible for 26% of the total volatility, appear to carry information, and this is a surprising result. From this point of view, it should be stressed that Markovitz's portfolio scheme, based on a purely historical determination of the correlation matrix, proves particularly weak, since the elements of the matrix itself are dominated by noise. Notwithstanding, simulations experiments with random matrices [28] show that, in the context of the classical portfolio problem (minimizing the portfolio variance under linear constraints) noise has relatively little effect. To leading order the solutions are determined by the stable, large eigenvalues, and the displacement of the solution due to noise is rather small. The picture is completely different, however, if we attempt to minimize the variance under non-linear constraint, like those we have in the problem of short-selling with margin account. In this problem the presence of noise in the correlation matrix leads to serious instability and a high degree of degeneracy of the solutions.

6 Conclusions

In this paper we have presented a model of portfolio optimization in the general case of futures markets. That is allowing for long buying/short selling of assets with a fixed margin requirement, and assuming a relatively high number of assets. In this perspective, this model generalizes some relevant results originally obtained by Lintner. The introduction of a nonlinear constraint in the Lagrangian function makes the optimization procedure to find the solution very difficult. Firstly, it is not possible to find a unique efficiency frontier because of the presence in the solution equation of a vector composed of a sequence of ± 1 (corresponding to buying or selling the single asset), whose order is undefined. The consequence is that we have a multiple equilibrium solution characterized by many local minima. The number of these minima is exponentially increasing function in the number of assets, like the number of metastable states in spin glasses. We have illustrated the analogies between the two models. In front of the problem of having many efficiency frontiers, all corresponding to a (local) minimum risk but not equivalent among them, we have implemented a successful numerical procedure to search for the minimum of all minima, in such a way to find a sort of *super-efficiency* frontier, that will allow to get the best portfolio in terms of relationship between risk and return. We applied our model to a concrete portfolio, formed by 16 assets chosen among the most traded ones at Nasdaq. We went through the whole two-step procedure, obtaining significant results, which may suggest further efforts of developing the model presented.

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References

1. H. Markovitz, *Portfolio selection: Efficient Diversification of Investments* (Wiley, New York, 1959).
2. B. Mandelbrot, J. Businness **36**, 394 (1963).
3. R.N. Mantegna, H.E. Stanley, *An Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge University Press, Cambridge UK, 2000).
4. J.-P. Bouchaud, M. Potters, *Theory of Financial Risk* (Alea-Saclay, Eyrolles, Paris, 1997).

5. E.J. Elton, M.J. Gruber, *Modern Portfolio Theory and Investment Analysis*, 5th edn. (Wiley, New York, 1995).
6. H. Levy, M. Sarnat, *Portfolio and Investment Selection: Theory and Practice* (Prentice-Hall International, London, 1984).
7. J. Lintner, Rev. Econon. Stat. **XLVII**, 13 (1965).
8. J. Lintner, J. Financ. Quantit. Analysis **VI**, 1173 (1971).
9. C. Azariadis, The problem of multiple equilibrium, unpublished, University of California, Los Angeles (1992).
10. M.R. Garey, D.S. Johnson, *Computers and Intractability: a Guide to the Theory of NP-Completness* (Freeman, San Francisco, 1979).
11. *Local Search in Combinatorial Optimization*, edited by E.H.L. Aarts, J.K. Lenstra (John Wiley & Sons Ttd., 1997).
12. S. Kirkpatrick, C.D. Gellat, M.P. Vecchi, Science **220**, 671 (1983).
13. S. Galluccio, J.P. Bouchaud, M. Potters, Physica A **259**, 449 (1998).
14. A. Gabor, I. Kondor, Physica A **274**, 222 (1999).
15. I. Kondor, Int. J. Theor. Appl. Finance **3**, 537 (2000).
16. B. Rosenau, V. Plerou, P. Gopikrishnan, H.E. Stanley, *Portfolio optimization and the random magnet problem*, e-print: `cond-mat/0111537` (2001).
17. M. Mezard, G. Parisi, M.A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
18. K.H. Fisher, J.A. Hertz, Spin Glasses (Cambridge University Press, 1991).
19. A.J. Bray, M.A. Moore, J. Phys. C **13**, L469 (1980).
20. F. Tanaka, S.F. Edwards, J. Phys. F **10**, 2769 (1980).
21. D.S. Dean, J. Phys. A **27**, L899 (1994).
22. C.O. Alexander, C.T. Leigh, J. Derivatives **4**, 50 (1997).
23. M.L. Mehta, *Random Matrices* (Academic Press, London, 1991).
24. L. Laloux, P. Cizeau, J.P. Bouchaud, M. Potters, Phys. Rev. Lett. **83**, 1467 (1999).
25. L. Laloux, P. Cizeau, J.P. Bouchaud, M. Potters, Risk **12**, 69 (1999).
26. V. Plerou, P. Gopikrishnan, B. Rosenow, L.A.N. Amaral, H.E. Stanley, Phys. Rev. Lett. **83**, 1471 (1999).
27. V. Plerou, P. Gopikrishnan, B. Rosenow, L.A.N. Amaral, T. Guhr, H.E. Stanley, e-print `cond-mat/0108023` (2001).
28. S. Pafka, I. Kondor, e-print `cond-mat/0111503` (2001).